

Dynamic Phasor Modeling and Low-frequency Oscillation Analysis of Single-phase Vehicle-grid System

Xiaoqin Lyu, Xiaoru Wang^{ID}, Senior Member, IEEE, Yulong Che, and Longyuan Li

Abstract—In practical operations, low-frequency oscillation (LFO) occurs and leads to converter blocking when multiple electrical rail vehicles at the platform are powered by the traction network. This paper proposes a small-signal model in state-space form for multiple vehicle-grid systems based on a dynamic phasor. This model uses the phasor amplitude and phase as variables to accurately describe the dynamics of the converter phase-domain control. An eigenvalue based-method is introduced to investigate the LFO with advantages of acquiring all oscillatory modes and analyzing participation factors. Two low-frequency dominant modes are identified by eigenvalues. Mode shape reveals that one of the modes involves the oscillations between the grid-connected converters and the traction network, and the other one involves the oscillations among these converters. Then the sensitivities of these two low-frequency modes to different system parameters are analyzed. Participation factors of system state variables, when the number of connected vehicle increases, are compared. Finally, the theoretical analysis is verified by nonlinear time-domain simulations and the modal analysis based on the estimation of signal parameters via the rotational invariance techniques (ESPRIT) method.

Index Terms—Dynamic phasor, eigenvalue analysis, electrical railway, grid-connected converter, low-frequency oscillation, participation factors.

I. INTRODUCTION

HIGH-speed railways have rapidly developed in China in recent years, with AC-DC-AC drive systems adopted by electrical rail vehicles (hereinafter referred to as vehicles). System instabilities in the electrical railway have occurred, including low-frequency oscillation (LFO), and harmonic instability [1], [2]. LFOs can be divided into two categories. One case first happened in a Norwegian railway, where the traction substations contained the rotary converters (each consisting

of a three-phase motor and a single-phase generator). This type of rotary converter is used to convert the three-phase 50 Hz supply voltage to the single-phase 16.5 Hz voltage. The 1.6 Hz oscillation related to the rotary converter occurred in this system [3]. The other case, without rotary converters in traction substations, happened in France and China [1], [4], which is studied in this paper. When multiple vehicles are stationed at the same spot and raised pantographs, with only auxiliary loads powered through the grid-connected converters, the LFO occurred. In the meantime, the traction motors did not work. The actual measured data and the related researches show that the phenomenon of this LFO is the amplitude and phase fluctuations of the voltages and currents along the traction network (the grid), and the voltages of the DC-link [5]. If the amplitude fluctuations exceed the protection threshold, the system may result in converter blocking and finally service delays.

The LFOs of electrical railways studied in this paper are an unstable phenomenon caused by the grid-connected converters of vehicles. There are two approaches for stability analysis of power converter-based systems. The first one is the impedance-based method in the frequency-domain [6]–[8]. In this method, the power converters and the source are modeled by frequency-domain impedances, and thus the stabilities can be analyzed by using generalized Nyquist criterion, bode diagram, etc. The main advantage of the impedance-based method is that system dynamics can be predicted without prior knowledge of the system internal parameters [8]. The second one is the eigenvalue-based method [9]. In the eigenvalue-based method, system stabilities are analyzed by the eigenvalues of the system state matrix. Compared with the impedance-based method, the advantages of the eigenvalue-based method are as follows. First, all oscillatory modes can be observed by eigenvalues, and eigenvalues indicate the oscillatory frequencies and damping ratios. Secondly, the influences of parameters on oscillatory frequencies and damping ratios can be analyzed simultaneously by the eigenvalue trajectories. Thirdly, the relative participation of each state variable in the corresponding modes can be obtained by participation factors, thus the critical states which have the greatest contribution to the modes are identified. Both of these two approaches have been widely used to evaluate the stabilities of power converter-based systems [10]–[18].

Manuscript received July 3, 2020; revised October 15, 2020; accepted October 25, 2020. Date of online publication November 20, 2020; date of current version March 17, 2021. This work was supported in part by the Fundamental Research Funds for the Central Universities of China (No. 2682019CX20), and in part by the Applied Basic Research Program of Science and Technology Plan Project of Sichuan Province of China (No. 2020YJ0252).

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DOI: 10.17775/CSEEJPES.2020.03160

There are several types of vehicles in the China railway system, and the grid-connected converters are single-phase converters. The control strategies of the grid-connected converters of different types of vehicles include the control in dq-domain and the control in phase-domain. One of the differences between these two control strategies lies in the current controllers. The input of current controllers in the dq-domain control is a time-invariant signal, while the input of the current controller in phase-domain control is a periodic time-varying signal. The modeling and the LFO analysis with grid-connected converters controlled in the dq-domain have been widely researched using the impedance-based method. The dq-domain impedances of these type of vehicles have been primarily studied in [1], [2], [17], [19], [20]. The influence of the number of vehicles and control parameters on the LFO, including voltage controller and current controller, etc. has been analyzed by means of calculating the poles of the system transfer function [1]. The improved forbidden-region criterions have been proposed to analyze the influence of the grid side impedance, the DC-link capacitance, voltage and current controller parameters on the LFO [17], [19]. Generalized Nyquist criterion has been applied to analyze the LFO [20]. The LFO has also been analyzed using the RLC circuit model of the vehicle-grid system in [21]. Moreover, eigenvalue-based method has been introduced to analyze the LFO caused by the interaction between the rotary converter in the substation and the vehicle [22].

However, there are fewer studies on the vehicle-grid system, whose grid-connected converters are controlled in the phase-domain. In the phase-domain control, the output of the voltage controller is a DC signal, while the input of the current controller is an AC signal. Therefore, the modeling of this control strategy will be different and particular. The impedance models, which only take into account the current controller dynamic, have been studied in [23], [24]. The influence of the number of vehicles and the controller sampling period on the LFO have been analyzed by means of the Nyquist criterion and bode diagram [23], and the influence of the catenary length on the LFO has been analyzed by using the bode diagram [24]. It has been proved that the voltage controller has a large effect on low-frequency stability, while the models in [23], [24] did not include the voltage controller. The voltage controller has been considered in the derivation of the transfer functions in [25], [26]. However, the phase dynamic of the AC side current was ignored during the modeling [25], [26]. Therefore, when analyzing the stability of the vehicle-grid system that controls the grid-connected converters in phase-domain, the existing impedance models or the transfer functions cannot accurately describe the system's different dynamics.

The vehicle-grid system is a time-variant and nonlinear system. Time-variance is due to the periodical operating trajectories of AC signals and their switching behavior. Nonlinearity results from the controllers. For the vehicle that applies phase-domain control, if the model is established in a stationary frame, variables of the AC side and the input of the current controller are time-variant at steady state, therefore the conventional linearization method cannot be used to obtain a linear and time-invariant model for small-signal stability analysis,

because a well-defined operating point cannot be identified. In addition, since the output of the voltage controller is the amplitude of the input AC signal of the current controller, it is difficult to directly build the model of all controllers in the dq-domain. In order to realize the time-invariance of the variables at steady state, and to accurately derive the model that can describe the different dynamics of the control in the phase-domain, a dynamic phasor modeling method has been proposed in [27]. This dynamic phasor modeling applies the transformation of a single-phase circuit from the stationary frame to the rotating frame to obtain the dynamic phasor, and uses the phasor amplitude and phase as variables to derive the differential equations of system dynamic elements, including equations of the traction network and the controllers, thus the time-invariance of equations can be realized. The output of the voltage controller is the amplitude of the input AC signal of the current controller, and the phasor is related with the amplitude and initial phase of the AC signal. Therefore, the advantage of using dynamic phasor modeling is not only to obtain a time-invariant model, but also to make the physical meaning of the variables in the small-signal model system clearer. A small-signal model of a single vehicle connected to the traction network based on a dynamic phasor has been analyzed in [27], which considers the complete structure and multiconductor nature of the traction network to realize the moving of the vehicle, and the high-frequency instability. However, unlike the high-frequency instability, when the LFO happens, the motors of the vehicles have not started, so that it is not necessary to consider the moving of the vehicles, and therefore the model can be greatly simplified.

The contributions of this paper are as follows. First, for grid-connected converters of vehicles which adopted phase-domain control and in view of the condition of the LFO occurrence, the small-signal model in state-space form of multiple vehicle-grid systems based on a dynamic phasor is proposed, which accurately describes the system's different dynamics. Secondly, two low-frequency dominant modes are identified from the proposed detailed small-signal model, and the oscillation nature of these two low-frequency modes is revealed by mode shape. Thirdly, sensitivities of these two low-frequency modes to different system parameters are analyzed and compared, and participation factors of state variables are introduced to find the critical elements that primarily affect the low-frequency stability.

The remainder of this paper is organized as follows. Section II presents the equivalent circuit of the multiple vehicle-grid system. In Section III, the system small-signal model is derived. In Section IV, eigenvalue trajectories under the changes of the number of vehicles connected to the grid, the controller parameters, the DC-link capacitance and the grid side impedance are investigated. In addition, the mode shape of two dominant modes and participation factors of the state variables are analyzed. In Section V, time-domain nonlinear simulations and modal analysis based on the estimation of signal parameters via the rotational invariance techniques (ESPRIT) method are performed to verify the eigenvalue analysis. Section VI presents the conclusions.

II. EQUIVALENT CIRCUIT OF VEHICLE-GRID SYSTEM

Figure 1 shows P_{veh} vehicles stay at the same spot and raise the pantographs. T represents the on-board transformer of the vehicle. L_{sb} , R_{sb} denote the equivalent inductance and resistance of the substation and related grid respectively, and u_{sb} is the supply voltage. In fact, each vehicle has not only one power unit (hereinafter called unit), but N units. The equivalent circuit of a single unit in which two paralleled single-phase grid-connected converters sharing one DC side is shown in Fig. 2. u_T is the input voltage of on-board transformers. L_T and R_T are the equivalent winding inductance and resistance of this transformer, respectively. C_d is the DC-link capacitance. L_2 and C_2 are the inductance and capacitance of the second harmonic filter. The subscript n represents the unit n .

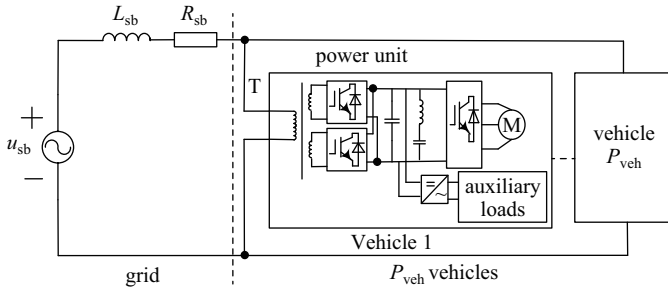


Fig. 1. Schematic diagram of the vehicle-grid system.

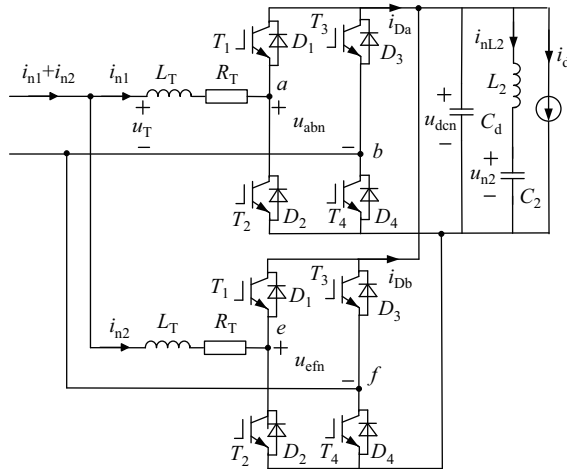


Fig. 2. Equivalent circuit of the unit (the grid-connected converters).

Figure 3 shows the corresponding grid side equivalent circuit with N units connected to the grid in parallel. u_s , R_s , and L_s are the parameters of the grid converted to the secondary side of the on-board transformer.

III. DYNAMIC PHASOR MODELING OF THE VEHICLE-GRID SYSTEM

A. Dynamic Phasor Modeling

A dynamic phasor was introduced based on the generalized averaging method in [28]. An AC non-sinusoidal signal x can

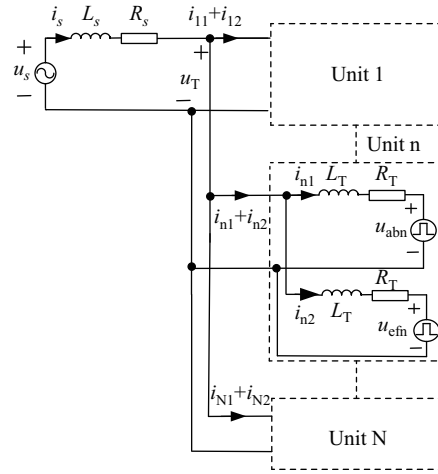


Fig. 3. Grid side equivalent circuit with N units in parallel.

be approximated with a Fourier series representation on the interval $\tau \in (t - T, t]$,

$$x(\tau) = \sum_k \langle x \rangle_k(t) e^{jk\omega_1\tau} \quad (1)$$

$$\langle x \rangle_k(t) = \frac{1}{T} \int_{t-T}^t x(\tau) e^{-jk\omega_1\tau} d\tau \quad (2)$$

where $\omega_1 = 2\pi/T$, $\langle x \rangle_k(t)$ is the Fourier coefficient.

The Fourier coefficient of the k th harmonic is the k th dynamic phasor, and the differential formula of k th dynamic phasor is [28]:

$$\frac{d\langle x \rangle_k(t)}{dt} = \left\langle \frac{dx}{dt} \right\rangle_k(t) - jk\omega_1 \langle x \rangle_k(t) \quad (3)$$

Equation (1) indicates that the k th harmonic component on the interval $(t - T, t]$ can be obtained by multiplying the k th dynamic phasor by $e^{jk\omega_1\tau}$, and conversely, the k th dynamic phasor can be obtained by transforming the k th harmonic component into the corresponding xy rotating frame. Therefore, the dynamic phasor modeling method can be described as, transform the system dominant frequency component into the phasor in the corresponding xy rotating frame, furthermore use the phasor to describe the dynamics of the system.

The traditional coordinate transformation is primarily used in three-phase systems. However, the vehicle is powered by a single-phase voltage, so that its coordinate transformation will be different. The Park transformation given by (4) is applied to convert fundamental signals in $\alpha\beta$ stationary frames into signals in xy fundamental frequency rotating frames. For a single-phase circuit, the real AC signal is usually set as the component in α frame, and the component in the β frame is a fictitious signal which lags the single-phase AC signal by 90° [29].

$$C_{2s/2r} = \begin{bmatrix} \cos \omega_1 t & \sin \omega_1 t \\ -\sin \omega_1 t & \cos \omega_1 t \end{bmatrix} \quad (4)$$

An AC voltage or current signal s at fundamental frequency ω_1 can be represented by:

$$s = s' \sin(\omega_1 t - \theta) = s' \cos(\omega_1 t - \theta - 90^\circ) \quad (5)$$

Convert s into xy fundamental frequency rotating frame,

$$\begin{cases} s_x = -s' \sin \theta \\ s_y = -s' \cos \theta \end{cases} \quad (6)$$

Then a vector $S \angle \sigma$ in xy frame is obtained,

$$\begin{cases} S = s' \\ \sigma = -90^\circ - \theta \end{cases} \quad (7)$$

From (5) and (7), it can be seen that when s is expressed by a cosine function, its phasor is equal to the vector in the corresponding rotating frame. Therefore, performing coordinate transformation on arbitrary AC signals by (4) obtains the phasor at the same frequency.

In addition, the derivatives of s_x and s_y with respect to time t are then represented by:

$$\begin{cases} \frac{ds_x}{dt} = -\sin \theta \frac{ds'}{dt} - s' \cos \theta \frac{d\theta}{dt} \\ \frac{ds_y}{dt} = -\cos \theta \frac{ds'}{dt} + s' \sin \theta \frac{d\theta}{dt} \end{cases} \quad (8)$$

Equation (8) indicates how the dynamics of the two orthogonal components in the xy rotating frame are described by the amplitude and phase of the phasor at the same frequency, so that the system dynamic phasor model can be derived.

B. Small-signal Model of AC and DC Sides of the Converter

To derive the small-signal model of the vehicle-grid system, some assumptions are made as follows.

- High order harmonics caused by PWM of the power electronics are neglected.
- Due to the second harmonic low-pass filter on the DC side, the secondary ripple of the DC-link voltage is neglected.

As shown in Fig. 2, i_{n1} and i_{n2} , u_{abn} and u_{efn} are the AC side currents and voltages of unit n , respectively. With the above assumptions, the two following formulas will make sense.

$$\begin{cases} u_{abn} = u_{efn} \\ i_{n1} = i_{n2} = i_n \end{cases} \quad (9)$$

Based on the average model over one switching period, voltage u_{abn} of unit n can be represented by:

$$u_{abn} = m_n \sin(\omega_1 t - \delta_n) u_{dcn} \quad (10)$$

where $m_n \sin(\omega_1 t - \delta_n)$ is the PWM modulating signal of unit n , m_n is the modulating index, and u_{dcn} is the DC-link voltage.

Assume that the supply voltage u_s is a sinusoidal signal at frequency ω_1 , that is $u_s = u'_s \sin \omega_1 t$. Therefore in Fig. 3, the AC side current i_n , the grid side current i_s and the input voltage u_T of the on-board transformers are also sinusoidal signals, which are represented as follows.

$$\begin{cases} i_n = i'_n \sin(\omega_1 t - \theta_n) \\ u_T = u'_T \sin(\omega_1 t - \phi_T) \\ i_s = i'_s \sin(\omega_1 t - \gamma) \end{cases} \quad (11)$$

The equations in time-domain including the AC side, DC side and the grid dynamics are given by:

$$L_T \frac{di_n}{dt} + R_T i_n = u_T - m_n \sin(\omega_1 t - \delta_n) u_{dcn} \quad (12)$$

$$L_s \frac{di_s}{dt} + R_s i_s = u_s - u_T \quad (13)$$

$$u_T = R_g \left(i_s - 2 \sum_{n=1}^N i_n \right) \quad (14)$$

$$C_d \frac{du_{dcn}}{dt} = 2m_n \sin(\omega_0 t - \delta_n) i_n - i_{nL2} - i_d \quad (15)$$

where R_g is an assumed virtual resistance between the ground and the pantographs of the vehicles. In order not to affect the eigenvalue analysis, R_g should be large enough [13]. i_{nL2} , i_d are the current of the second harmonic filtering branch and load current of the DC side, respectively.

Transform (12), (13), and (14) into xy rotating frames, and the components in the xy frame and their derivatives are represented by expressions of phasor amplitude and phase (given in the appendix for detailed procedures). For (15), since the second harmonic current of the DC side primarily passes through the filtering branch, the secondary ripple of the DC-link voltage u_{dcn} is not considered. After linearizing equations around the equilibrium point, the small-signal models of the AC and DC sides of unit n are given by:

$$\begin{cases} L_T \frac{d\Delta i'_n}{dt} = \Delta u'_T - L_T \omega_1 I_{n0} \Delta \theta_n - R_T \Delta i'_n - U_{dc0} \Delta m_n - M_{n0} \Delta u_{dc} \\ L_T I_{n0} \frac{d\Delta \theta_n}{dt} = -R_T I_{n0} \Delta \theta_n + L_T \omega_1 \Delta i'_n + U_{T0} \Delta \phi_T - M_{n0} U_{dc0} \Delta \delta_n \end{cases} \quad (16)$$

$$\begin{cases} L_s \frac{d\Delta i'_s}{dt} = -R_s \Delta i'_s - L_s \omega_1 I_{s0} \Delta \gamma - \Delta u'_T + \cos \gamma_0 \Delta u'_s \\ L_s I_{s0} \frac{d\Delta \gamma}{dt} = -R_s I_{s0} \Delta \gamma + L_s \omega_1 \Delta i'_s - U_{T0} \Delta \phi_T - \sin \gamma_0 \Delta u'_s \end{cases} \quad (17)$$

$$\begin{cases} \Delta u'_T = R_g \Delta i'_s - 2R_g \sum_{n=1}^N \Delta i'_n \\ U_{T0} \Delta \phi_T = R_g I_{s0} \Delta \gamma - 2R_g \sum_{n=1}^N I_{n0} \Delta \theta_n \end{cases} \quad (18)$$

$$C_d \frac{d\Delta u_{dcn}}{dt} = I_{n0} \Delta m_n + M_{n0} \Delta i'_n - \Delta i_d \quad (19)$$

where I_{n0} , θ_{n0} , U_{dc0} , M_{n0} , δ_{n0} , I_{s0} , γ_0 , U_{T0} and ϕ_{T0} are the steady state values, and $\theta_{n0} = \phi_{T0} = \gamma_0 \approx \delta_{n0}$. $\Delta i'_n$, $\Delta \theta_n$, Δu_{dcn} , Δm_n , $\Delta \delta_n$, $\Delta i'_s$, $\Delta \gamma$, Δu_T and $\Delta \phi_T$ are the small-signal perturbations.

Equation (17) can also be represented by:

$$\frac{d}{dt} \begin{bmatrix} \Delta i'_s \\ \Delta \gamma \end{bmatrix} = \mathbf{Q} \begin{bmatrix} \Delta i'_s \\ \Delta \gamma \end{bmatrix} + \mathbf{M} \begin{bmatrix} \Delta u'_T \\ \Delta \phi_T \end{bmatrix} + \mathbf{C}_4 \begin{bmatrix} \Delta u'_s \\ \Delta i_d \end{bmatrix} \quad (20)$$

where $\mathbf{Q}$, $\mathbf{M}$ and $\mathbf{C}_4$ are the corresponding coefficient matrices.

C. The Controller

Figure 4 shows the phase-domain control of the single-phase grid-connected converter, which includes a voltage controller

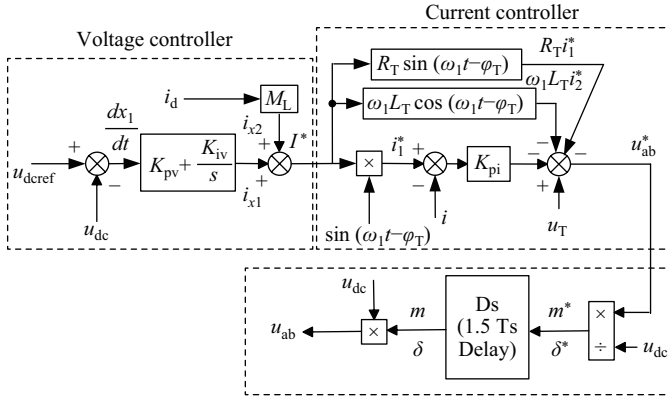


Fig. 4. Phase-domain control of the single-phase grid-connected converter.

and a current controller. The output of the voltage controller is the amplitude of the AC side reference current of the converter. The output of the current controller is the AC side reference voltage of the converter.

The dynamic of the voltage controller is given by:

$$\frac{dx_{1n}}{dt} = u_{dcref} - u_{dcn} \quad (21)$$

Linearize (21) around the equilibrium point,

$$\frac{d\Delta x_{1n}}{dt} = -\Delta u_{dcn} \quad (22)$$

The AC side reference voltage in time-domain obtained from the output of the current controller is represented by:

$$u_{abn}^* = u_T - R_T i_{1n}^* - \omega_1 L_T i_{2n}^* - K_{pi}(i_{1n}^* - i_n) \quad (23)$$

The reference amplitude I_n^* of AC side current and its time-domain instantaneous value in (23) are given by:

$$\begin{cases} I_n^* = K_{pv}(u_{dcref} - u_{dcn}) + K_{iv}x_{1n} + M_L i_d \\ i_{1n}^* = I_n^* \sin(\omega_1 t - \phi_T) \\ i_{2n}^* = I_n^* \cos(\omega_1 t - \phi_T) \end{cases} \quad (24)$$

where $\sin(\omega_1 t - \phi_T)$, $\cos(\omega_1 t - \phi_T)$ are the sine and cosine signals in phase with the input voltage u_T , respectively. M_L is the ratio of the AC side current amplitude to the DC side current under the steady state.

Then the modulation signal can be obtained by:

$$m_n^* \sin(\omega_1 t - \delta_n^*) = \frac{u_{abn}^*}{u_{dcn}} \quad (25)$$

Substitute (23) into (25), and perform a coordinate transformation. Then represent the components in the xy frame and their derivatives by the expressions of the phasor amplitude and phase. After linearizing it around the equilibrium point, the small-signal models of the modulation signal are represented by:

$$\begin{aligned} U_{dc0} \Delta m_n^* &= \Delta u_T' - \omega_1 L_T I_{n0}^* \Delta \phi_T + \\ &K_{pi} \Delta i_n - (R_T + K_{pi}) K_{iv} \Delta x_{1n} - \\ &(R_T + K_{pi}) M_L \Delta i_d + \\ &((R_T + K_{pi}) K_{pv} - M_{n0}) \Delta u_{dcn} \\ M_{n0} U_{dc0} \Delta \delta_n^* &= U_{T0} \Delta \phi_T + \omega_1 L_T K_{iv} \Delta x_{1n} - \end{aligned} \quad (26)$$

$$\begin{aligned} &\omega_0 L_T K_{pv} \Delta u_{dcn} - (R_T + K_{pi}) I_{n0}^* \Delta \phi_T + \\ &\omega_0 L_T M_L \Delta i_d + K_{pi} I_{n0} \Delta \theta_n \end{aligned} \quad (27)$$

Equation (26) and (27) can be expressed as:

$$\begin{bmatrix} \Delta m_n^* \\ \Delta \delta_n^* \end{bmatrix} = E_1 \Delta x_n + F \begin{bmatrix} \Delta u_T' \\ \Delta \phi_T \end{bmatrix} + J \begin{bmatrix} \Delta u_s' \\ \Delta i_d \end{bmatrix} \quad (28)$$

where $\Delta x_n = [\Delta x_{1n} \ \Delta u_{dcn} \ \Delta i_n' \ \Delta \theta_n]^T$, E_1 , F , and J are the corresponding coefficient matrices.

A digital controlled system has one sampling period T_s computation delay and $0.5 T_s$ time delay due to the pulse-width modulation [30]. The transfer function D_s of $1.5 T_s$ time delay shown in Fig. 4 is expressed in [31].

$$D_s \approx \frac{1 - 0.5 \cdot s \cdot T_s}{(1 + 0.5 \cdot s \cdot T_s)^2} \quad (29)$$

Equation (29) can be represented by a state-space equation using state variables x_{d1n} ($x_{d1n} = [x_{d11n} \ x_{d12n}]^T$) and x_{d2n} ($x_{d2n} = [x_{d21n} \ x_{d22n}]^T$).

$$\frac{d\Delta x_{dn}}{dt} = G \Delta x_{dn} + H \Delta v_n^* \quad (30)$$

$$\Delta v_n = I \Delta x_{dn} \quad (31)$$

where

$$\begin{aligned} \Delta x_{dn} &= [\Delta x_{d11n} \ \Delta x_{d12n} \ \Delta x_{d21n} \ \Delta x_{d22n}]^T, \\ G &= \begin{bmatrix} G_1 & 0 \\ 0 & G_1 \end{bmatrix}, H = \begin{bmatrix} H_1 & 0 \\ 0 & H_1 \end{bmatrix}, I = \begin{bmatrix} I_1 & 0 \\ 0 & I_1 \end{bmatrix}, \\ H_1 &= [0 \ 1/(0.25T_s^2)]^T, I_1 = [1 \ -0.5T_s], \\ \Delta v_n^* &= [\Delta m_n^* \ \Delta \delta_n^*]^T, \Delta v_n = [\Delta m_n \ \Delta \delta_n]^T, \\ G_1 &= \begin{bmatrix} 0 & 1 \\ -\frac{1}{0.25T_s^2} & -\frac{1}{0.25T_s} \end{bmatrix}. \end{aligned}$$

D. Small-signal Model of Unit N Without the Grid Dynamic

First, when combining (16), (19), and (22), the small-signal model of unit n , without the grid dynamic and the time delay, is represented by:

$$\frac{d\Delta x_n}{dt} = A_1 \Delta x_n + D_1 \begin{bmatrix} \Delta u_T' \\ \Delta \phi_T \end{bmatrix} + C_1 \begin{bmatrix} \Delta m_n \\ \Delta \delta_n \end{bmatrix} + B_1 \Delta u \quad (32)$$

where $\Delta u = [\Delta u_s' \ \Delta i_d]^T$, A_1 , B_1 , C_1 and D_1 are the corresponding coefficient matrices.

By combination with (30), (31) and (32), the small-signal model of unit n considering the time-delay is given by:

$$\begin{aligned} \frac{d\Delta x_{LSCn}}{dt} &= A_2 \Delta x_{LSCn} + D_2 \begin{bmatrix} \Delta u_T' \\ \Delta \phi_T \end{bmatrix} + C_2 \begin{bmatrix} \Delta m_n \\ \Delta \delta_n \end{bmatrix} \\ &+ H_2 \begin{bmatrix} \Delta m_n^* \\ \Delta \delta_n^* \end{bmatrix} + B_2 \Delta u \end{aligned} \quad (33)$$

$$\begin{bmatrix} \Delta m_n \\ \Delta \delta_n \end{bmatrix} = I_2 \Delta x_{LSCn} \quad (34)$$

where $\Delta x_{LSCn} = [\Delta x_n \ \Delta x_{dn}]^T$, $B_2 = [B_1 \ 0]^T$, $C_2 = [C_1 \ 0]^T$, $D_2 = [D_1 \ 0]^T$, $I_2 = [0 \ I]$, $A_2 = \begin{bmatrix} A_1 & 0 \\ 0 & G \end{bmatrix}$, $H_2 = \begin{bmatrix} 0 \\ H \end{bmatrix}$.

Substitute (28) and (34) into (33),

$$\frac{d\Delta x_{\text{LSCn}}}{dt} = (A_2 + C_2 I_2 + H_2 E_2) \Delta x_{\text{LSCn}} + (D_2 + H_2 F) \begin{bmatrix} \Delta u'_T \\ \Delta \phi_T \end{bmatrix} + (H_2 J + B_2) \Delta u \quad (35)$$

where $E_2 = [E_1 \ 0]$.

Let $A_3 = (A_2 + C_2 I_2 + H_2 E_2)$, $D_3 = (D_2 + H_2 F)$, and $B_3 = (H_2 J + B_2)$, then the small-signal model of unit n is given by:

$$\frac{d\Delta x_{\text{LSCn}}}{dt} = A_3 \Delta x_{\text{LSCn}} + D_3 \begin{bmatrix} \Delta u'_T \\ \Delta \phi_T \end{bmatrix} + B_3 \Delta u \quad (36)$$

E. Small-signal Model of P_{veh} Vehicles Connected to the Grid

When there are P_{veh} vehicles connected to the grid, a total of N units are powered. Using (20) and (36), the small-signal model for a multiple vehicle-grid system including N units and the grid dynamic is given by:

$$\frac{d\Delta x_{\text{sys}}}{dt} = A_4 \Delta x_{\text{sys}} + D_4 \begin{bmatrix} \Delta u'_T \\ \Delta \phi_T \end{bmatrix} + B \Delta u \quad (37)$$

where

$$\Delta x_{\text{sys}} = [\Delta x_{\text{LSC1}} \ \Delta x_{\text{LSC2}} \ \dots \ \Delta x_{\text{LSCN}} \ \Delta i'_s \ \Delta \gamma]^T$$

$$A_4 = \begin{bmatrix} A_3 & & 0 \\ & \dots & \\ 0 & & A_3 \\ & & Q \end{bmatrix}, D_4 = [D_3 \ \dots \ D_3 \ M]^T$$

$$B = [B_3 \ \dots \ B_3 \ C_4]^T$$

The input voltage of on-board transformer can be represented by:

$$\begin{bmatrix} \Delta u'_T \\ \Delta \phi_T \end{bmatrix} = D_5 \Delta x_{\text{sys}} \quad (38)$$

where

$$E_3 = [0 \ 0 \ -2R_g \ 0 \ 0 \ 0 \ 0 \ 0], \ E_4 = [R_g \ 0],$$

$$E_5 = \begin{bmatrix} 0 & 0 & 0 & \frac{-2R_g I_{\text{no}}}{U_{T0}} & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$E_6 = \begin{bmatrix} 0 & \frac{R_g I_{s0}}{U_{T0}} \end{bmatrix}, \ D_5 = \begin{bmatrix} E_3 & \dots & E_3 & E_4 \\ E_5 & \dots & E_5 & E_6 \end{bmatrix}$$

Substitute (38) into (37), then the small-signal model of P_{veh} vehicles connected to the grid is given by:

$$\frac{d\Delta x_{\text{sys}}}{dt} = A \Delta x_{\text{sys}} + B \Delta u \quad (39)$$

where $A = (A_4 + D_4 D_5)$.

IV. LFO ANALYSIS BASED ON EIGENVALUES

This paper analyses the real system of the China railway HXD1 vehicle. The main circuit parameters are shown in Table I. HXD1 has two on-board transformers, each with four secondary windings. Hence there are four units in one vehicle. For the voltage controller, the proportional gain K_{pv} is 0.1, and the integral gain K_{iv} is 13. For the current controller, the proportional gain K_{pi} is 0.1. The controller sampling frequency is 8 kHz.

TABLE I
PARAMETERS OF THE MAIN CIRCUIT

Parameter	Value
Grid side inductance L_s	0.92 mH
Resistance of on-board transformer R_T	0.1 Ω
Inductance of on-board transformer L_T	1.8 mH
Supply voltage u_s	970 V, 50 Hz
DC-link capacitance C_d	12 mF
DC-link reference voltage u_{dref}	1850 V
Load current i_d	20 A
Switching frequency f_r	250 Hz

A. Oscillatory Modes of a Single Vehicle Connected to the Grid

Eigenvalue analysis is a method to obtain the oscillatory modes by calculating the eigenvalues of the state matrix, which indicate the oscillatory frequencies and damping ratios. Calculating the eigenvalues of the state matrix A of (39), the oscillatory modes of the vehicle-grid system can be identified.

The participation factor shows the relative participation of the state in the corresponding mode. p_{mi} denote the participation factor of the m -th state variable in the i -th mode of the system, and p_{mi} is actually equal to the sensitivity of eigenvalue λ_i to the element a_{mm} of the system state matrix [9].

$$p_{mi} = \frac{\partial \lambda_i}{\partial a_{mm}} \quad (40)$$

Participation factors can be calculated using the left and right eigenvectors. Participation matrix P is represented by:

$$P = [P_1 \ P_2 \ \dots \ P_m] \quad (41)$$

where $P_i = [p_{1i} \ p_{2i} \ \dots \ p_{mi}]^T = [\varphi_{1i} \psi_{i1} \ \varphi_{2i} \psi_{i2} \ \dots \ \varphi_{mi} \psi_{im}]^T$, ψ_i and φ_i is the left and right eigenvectors associated with the eigenvalue λ_i , respectively. ψ_{im} and φ_{mi} are the m -th items of the left and right eigenvectors, respectively.

The right eigenvector φ_i of the eigenvalue λ_i , called the mode shape, indicates the observabilities of the system state variables to the corresponding mode. The elements of the right eigenvector φ_i are complex numbers, which can be expressed by amplitude and phase. The m -th element of the right eigenvector φ_i gives the activity of the variable x_m in the mode. The element with the largest amplitude of the right eigenvector is taken as the reference amplitude, and the phase is taken as the reference phase. Then the normalized amplitude of each element of the right eigenvector can be computed by dividing the original amplitude by the reference amplitude, and the normalized phase can be computed by subtracting the reference phase from the original phase.

Figure 5 shows the oscillatory modes when a single vehicle is connected to the grid. There are one pair of complex eigenvalues, corresponding to mode λ_{1-2} , and three pairs of repetitive complex eigenvalues, corresponding to mode λ_{3-4} , both of which are close to the imaginary axis. It can be concluded that, modes λ_{1-2} and λ_{3-4} are the dominant modes, and the oscillation frequencies are both below 5 Hz.

Analysis of the mode shape helps to further explore the nature of modes λ_{1-2} and λ_{3-4} . For all state variables of each unit and the grid side, the corresponding elements of the

right eigenvector are shown in Table II and Fig. 6. It can be concluded that, mode λ_{1-2} exists in the state variables of all units and the grid side, which indicates that mode λ_{1-2} can be observed on both the units and the grid side. Moreover, the grid side has the strongest observability of mode λ_{1-2} .

TABLE II
MODE SHAPE CORRESPONDING TO MODE λ_{1-2}

State variable	Amplitude	Phase ($^\circ$)
Unit 1		
DC-link voltage	0.2861	-93.1027
AC side current	0.1250	0
Unit 2		
DC-link voltage	0.2861	-93.1028
AC side current	0.1250	0
Unit 3		
DC-link voltage	0.2861	-93.1028
AC side current	0.1250	0
Unit 4		
DC-link voltage	0.2861	-93.1028
AC side current	0.1250	0
Grid side current	1	0
Remaining states	< 0.02	—

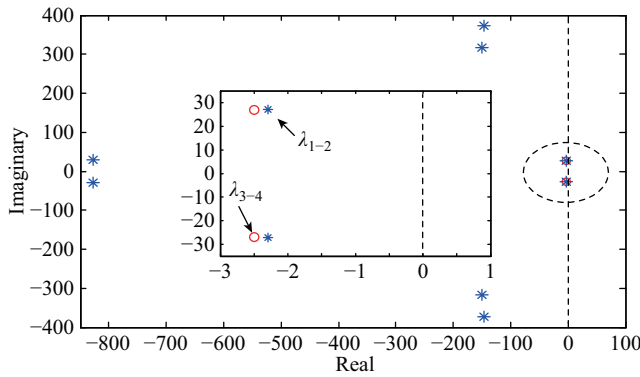


Fig. 5. Oscillatory modes of one vehicle connected to the grid ($K_{pv} = 0.1$, $K_{iv} = 13$, $K_{pi} = 0.1$) ($P_{veh} = 1$).

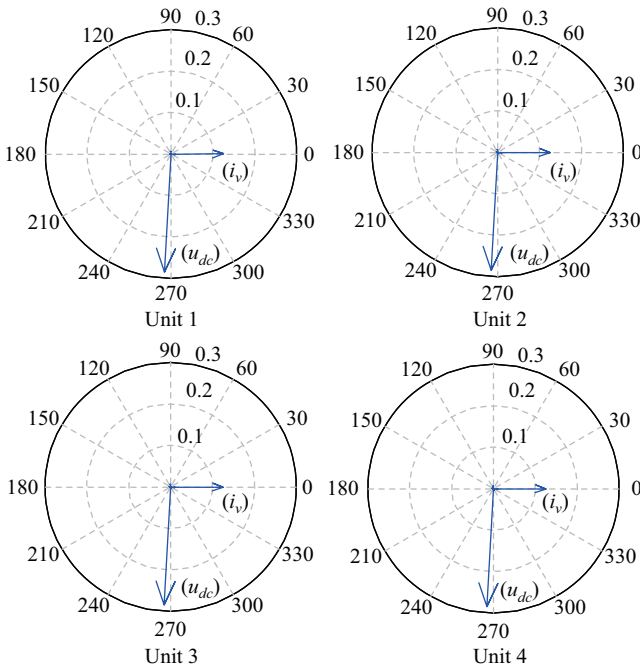


Fig. 6. Mode shape of state variables of four units corresponding to mode λ_{1-2} .

For the same state variables of each unit, the corresponding elements of the right eigenvector have the same amplitude and phase, indicating that when the oscillation is observed on this variable, the oscillating amplitude and phase of different units are exactly the same, or different units have the same observabilities of mode λ_{1-2} . The above analysis reflects that, for the oscillation characteristics of mode λ_{1-2} , all units act as a whole to interact with the grid side. So mode λ_{1-2} can be regarded as the in-phase mode [32].

Figure 7 shows the mode shape of unit state variables corresponding to mode λ_{3-4} . For each unit, the elements with relatively large amplitude are displayed, and the other elements cannot be clearly shown in the figure because the amplitude is relatively too small. It should be noted that, for the grid side state variable, the amplitude of the corresponding element of the right eigenvector of mode λ_{3-4} is zero, which is not drawn in the figure, indicating that the grid side can hardly observe the mode λ_{3-4} . It means the grid side does not participate in this oscillation. In addition, for the same state variables of each unit, the amplitude and phase of the corresponding elements of the right eigenvector are not equal. The above analysis reflects that mode λ_{3-4} is an oscillation existing among these units, and has no relation to the grid side. So mode λ_{3-4} can be regarded as the anti-phase mode.

B. Low-frequency Modes of Multiple Vehicles Connected to the Grid

1) Eigenvalue Trajectories

When the number of vehicles connected to the grid gradually increases ($P_{veh} = 1 \sim 10$), the number of units increases at a rate of $4 \times P_{veh}$ times. The system model is updated and the state matrix eigenvalues are recalculated. A pair of complex eigenvalues, corresponding to mode λ_{1-2} , and $(4P_{veh} - 1)$ pairs of repetitive complex eigenvalues, corresponding to mode λ_{3-4} are observed. The trajectories of modes λ_{1-2} and λ_{3-4} change as shown in Fig. 8. The eigenvalues gradually move towards the imaginary axis, which indicates that as the number of vehicles connected to the grid increases, the damping ratios of these modes are reduced. Especially when the number of vehicles increases to 9, mode λ_{1-2} enters the right half plane, and negative damping occurs. In addition, mode λ_{3-4} are always located in the left half-plane, and the damping is always positive. Compared to mode λ_{3-4} , mode λ_{1-2} is more sensitive to the number of vehicles. The increase in the number of vehicles causes the damping of mode λ_{1-2} to be significantly reduced, which is the reason that the system enters an unstable state.

To clearly show how the damping ratios and oscillatory frequencies of mode λ_{1-2} are affected by the number of vehicles, the eigenvalues of mode λ_{1-2} are given in Table III when the number of vehicles are 7, 8, and 9, respectively. Particularly, the oscillatory frequency and damping ratio are 4.01 Hz and -0.001 respectively when 9 vehicles are connected.

According to the previous modal shape analysis, mode λ_{3-4} is the oscillation among the units, and the grid side does not participate in this oscillation, while mode λ_{1-2} is reflected in both the grid side and the unit state variables, and the grid side has the strongest observability of this mode. The increase

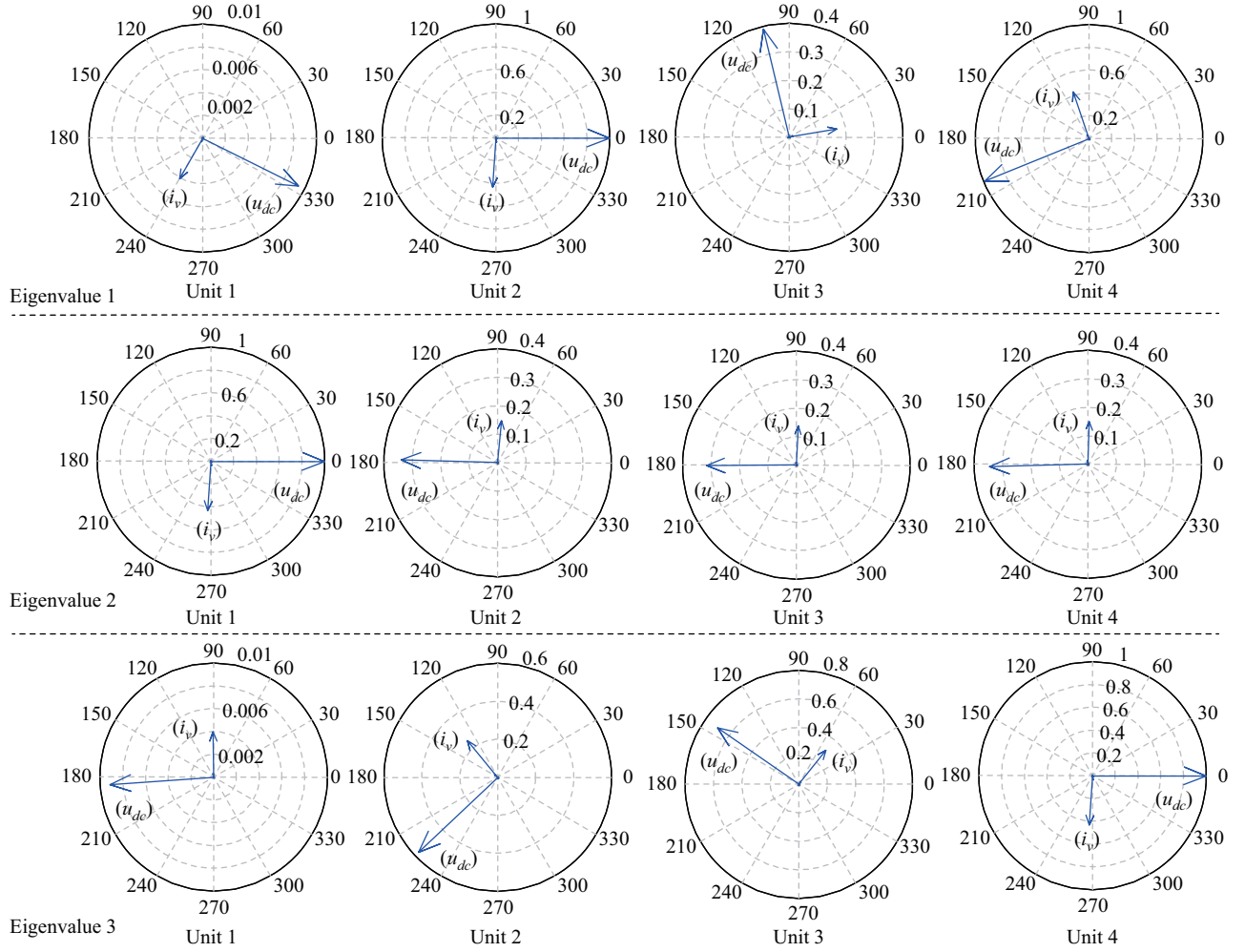


Fig. 7. Mode shape of state variables of four units corresponding to mode λ_{3-4} .

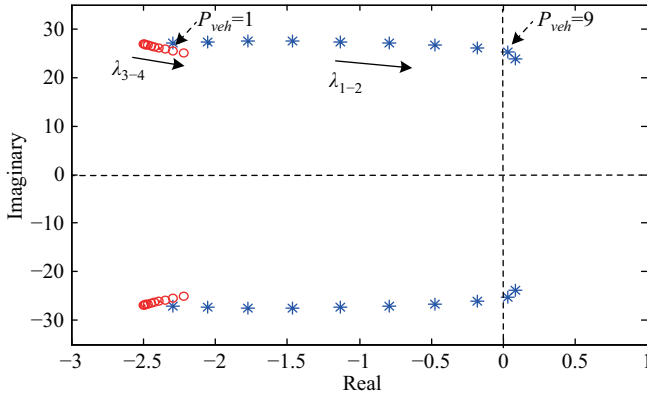


Fig. 8. Eigenvalue trajectories as the number of vehicles connected to the grid increases ($1 \leq P_{veh} \leq 10$, step:1) ($K_{pv} = 0.1$, $K_{iv} = 13$, $K_{pi} = 0.1$).

in the number of vehicles causes negative damping of mode λ_{1-2} so that LFO occurs. If only mode λ_{1-2} is analyzed, from the perspective of the consistency and integrity of all units during oscillating, the equivalent model of a single unit of a vehicle connected to the grid can be used. In order to keep the AC side current of this single unit unchanged, and to keep the unit input voltage the same, the network impedance

must be multiplied by $4P_{veh}$. Table IV shows the eigenvalues calculated based on this equivalent model, which are consistent with the data from the detailed model in Table III. It indicates that the equivalent model completely retains the oscillation characteristics of mode λ_{1-2} when the number of vehicles increases. Therefore, the low-frequency oscillation caused by the increased number of the connected vehicles is equivalent to the system's unstable nature of a single unit connected to the weak grid.

TABLE III
EIGENVALUES WHEN THE NUMBER OF VEHICLES INCREASES

Mode	Number of vehicles	Eigenvalues	Frequency (Hz)	Damping ratio
λ_{1-2}	7	$-0.472 \pm j26.744$	4.26	0.018
	8	$-0.185 \pm j26.141$	4.16	0.007
	9	$0.0294 \pm j25.234$	4.01	-0.001

TABLE IV
EIGENVALUES OF EQUIVALENT MODEL

Mode	Multiple of network impedance	Eigenvalue	Frequency (Hz)	Damping ratio
λ_{1-2}	$4P_{veh} = 4 \times 7$	$-0.472 \pm j26.742$	4.26	0.018
	$4P_{veh} = 4 \times 8$	$-0.185 \pm j26.139$	4.16	0.007
	$4P_{veh} = 4 \times 9$	$0.0287 \pm j25.232$	4.01	-0.001

2) Participation Factors

For mode λ_{1-2} , participation factors of the state variables are listed in Table V. Obviously, the grid side current i'_s , the state of voltage controller x_{1n} and the DC-link voltage u_{dcn} have larger participation factors, which means mode λ_{1-2} is highly sensitive to the state variables of the grid, the voltage controller, and the DC-link.

TABLE V
PARTICIPATION ANALYSIS OF MODE

State ($n = 1, \dots, N$)	Participation factor			
	Six vehicles	Seven vehicles	Eight vehicles	Nine vehicles
i'_s	0.0125	0.0152	0.0182	0.0218
x_{1n}	0.0207	0.0177	0.0154	0.0136
u_{dcn}	0.0210	0.0179	0.0157	0.0139
x_{d21n}	0.0030	0.0030	0.0030	0.0028

Remaining states < 0.0025

The vehicles connected to the grid are the same type, and the participation factors of the same state variables of the units have the same values. As the number of vehicles increases, the participation factors of the state variables of the voltage controller and the DC-link in a single unit are seen to gradually decrease, however, taking into account all units connected to the grid, the sum of participation factors ($N \times p_{mi}$) of either one of these two state variables almost remains unchanged. For instance, when six vehicles are connected to the grid, there are twenty-four units, then the sum of participation factors of u_{dcn} is 0.5040 (0.0210×24), which is almost equal to the sum of that when seven vehicles are connected to the grid. In other words, when considering the state variables of the voltage controllers and DC-link of all units, the sum of the relative participations of these state variables will hardly be affected by the number of vehicles. In addition, the state variables of the voltage controllers and DC-link of all units are the most critical elements that affect the system low-frequency stability. When the number of vehicles P_{veh} increases, the participation factor of the grid side current increases, indicating that for mode λ_{1-2} , the greater the number of vehicles, the higher the relative participation of the grid state variable.

C. Influences of Parameters on the Low-frequency Modes

When analyzing the influences of the controller and circuit parameters on the low-frequency modes, assume that there are 9 vehicles connected to the grid, and the initial parameters are the same as those in Fig. 8.

1) Influence of the Voltage Controller

In order to analyze the influences of the voltage controller parameters on modes λ_{1-2} and λ_{3-4} , the proportional gain K_{pv} and integral gain K_{iv} are respectively increased. Fig. 9 shows the eigenvalue trajectories as K_{pv} and K_{iv} increase. It can be seen that as K_{pv} increases, shown in Fig. 9(a), damping ratios of modes λ_{1-2} and λ_{3-4} gradually increase, especially mode λ_{1-2} moves from the right half-plane to the left half-plane. When K_{pv} is equal to 0.12, damping of mode λ_{1-2} becomes positive, therefore the system enters a stable state. As K_{iv} increases, shown in Fig. 9(b), modes λ_{1-2} and λ_{3-4} gradually move to the imaginary axis. When K_{iv} is below 12, damping of mode λ_{1-2} remains positive, which makes the system stable.

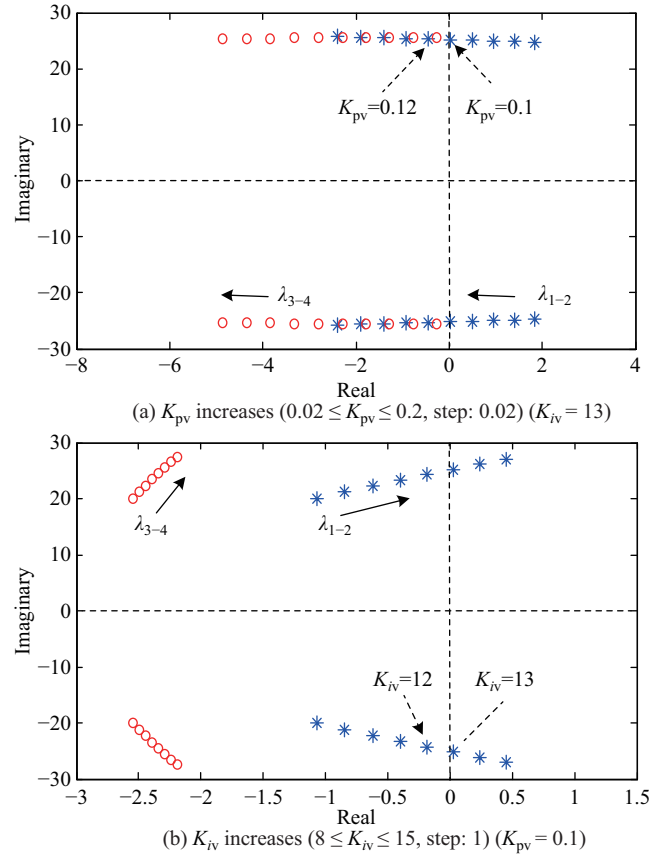


Fig. 9. Eigenvalue trajectories as the voltage controller parameters change ($K_{pi} = 0.1$) ($P_{veh} = 9$).

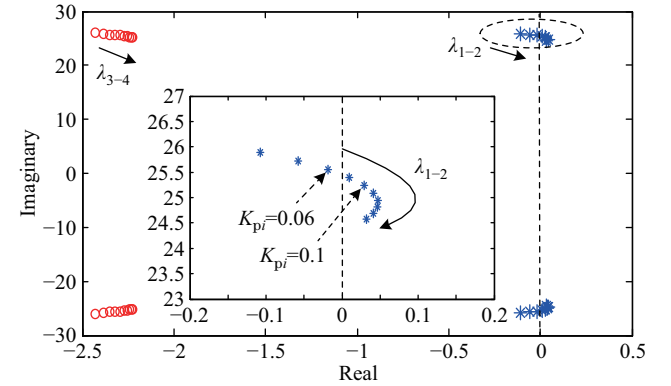


Fig. 10. Eigenvalue trajectories as the current controller proportional gain K_{pi} increases ($0.02 \leq K_{pi} \leq 0.2$, step: 0.02) ($K_{pv} = 0.1$, $K_{iv} = 13$) ($P_{veh} = 9$).

2) Influence of the Current Controller

Eigenvalue trajectories of mode λ_{1-2} and λ_{3-4} as the current controller proportional gain K_{pi} increases are shown in Fig. 10. Mode λ_{3-4} remains in the left half-plane, while mode λ_{1-2} gradually moves from the left half-plane to the right half-plane. When K_{pi} is below 0.06, damping of mode λ_{1-2} are positive, which makes the system stable. In addition, comparing with the eigenvalue trajectories shown in Fig. 9(a), these two modes are both less sensitive to K_{pi} . That is, mode λ_{1-2} and λ_{3-4} are more sensitive to the voltage controller parameters.

3) Influence of Grid Side Impedance

Grid side impedance includes the equivalent inductances of the substation and the traction network. When the grid side inductance increases in the condition that 9 vehicles are connected, the eigenvalue trajectories of modes λ_{1-2} and λ_{3-4} are shown in Fig. 11. It can be observed that as the value of grid side inductance increases, the damping ratios of modes λ_{1-2} and λ_{3-4} become smaller, especially mode λ_{1-2} gradually moves to the right half-plane. Moreover, mode λ_{1-2} is more sensitive to the grid side inductance than mode λ_{3-4} .

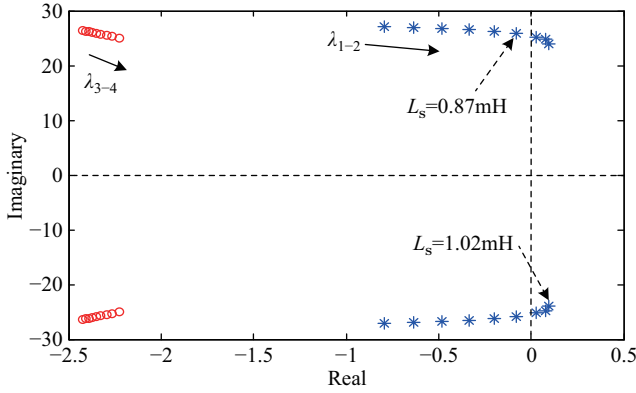


Fig. 11. Eigenvalue trajectories as grid side inductance increases (0.62 mH $\leq L_s \leq 1.02$ mH, step: 0.05 mH) ($K_{pv} = 0.1$, $K_{iv} = 13$, $K_{pi} = 0.1$) ($P_{veh} = 9$).

4) Influence of DC-link Capacitance

The value of DC-link capacitance C_d of the units also affects the eigenvalues, as shown in Fig. 12. As C_d increases, the oscillatory frequencies of modes λ_{1-2} and λ_{3-4} are significantly reduced. For damping of the modes, mode λ_{3-4} is more sensitive to C_d . For oscillatory frequency of mode λ_{1-2} , when C_d is 12 mF, the oscillatory frequency is 4.01 Hz, which is given in Table III. When C_d is 16 mF, the system is still unstable, however, the oscillatory frequency of mode λ_{1-2} reduces to 3.48 Hz.

The damping changes with the parameters are concluded as follows. The number of vehicles has a great influence on mode λ_{1-2} , while mode λ_{3-4} has low sensitivity to the number

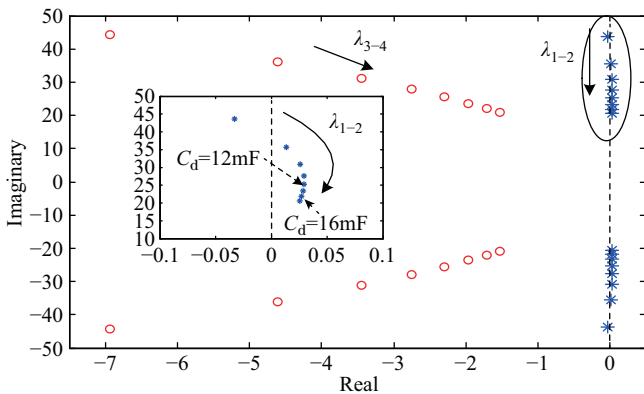


Fig. 12. Eigenvalue trajectories as DC-link capacitance increases (4 mF $\leq C_d \leq 18$ mF, step: 2 mF) ($K_{pv} = 0.1$, $K_{iv} = 13$, $K_{pi} = 0.1$) ($P_{veh} = 9$).

of vehicles. When the grid side L_s changes in the range of 0.62 mH to 1.02 mH, 67%~111% of the original value, the damping of modes λ_{1-2} and λ_{3-4} changes in the ranges of 0.029 to -0.004 and 0.091 to 0.089, respectively. When DC-link capacitance C_d changes in the range of 8 mF to 14 mF, 67%~117% of the original value, the damping of modes λ_{1-2} and λ_{3-4} changes in the ranges of -0.0008 to -0.001 and 0.109 to 0.083, respectively. Moreover, as shown in Fig. 12, the smaller the C_d , the greater the influence on the damping of mode λ_{3-4} . When K_{pv} of voltage controller changes in the range of 0.06 to 0.12, the range of 60% ~ 120% of the original value, the damping of modes λ_{1-2} and λ_{3-4} changes in the ranges of -0.038 to 0.017 and 0.05 to 0.109, respectively. When K_{pi} of the current controller changes in the same range, 0.06 to 0.12, the damping of mode λ_{1-2} and λ_{3-4} changes in the ranges of 0.0007 to -0.002 and 0.091 to 0.089, respectively. When K_{iv} of the voltage controller changes in the range of 8 to 15, that is, the range of 62%~115% of the original value, the damping of mode λ_{1-2} and λ_{3-4} change in the ranges of 0.053 to -0.017 and 0.127 to 0.079, respectively. Therefore, it can be concluded that, damping of mode λ_{1-2} has high sensitivity to L_s but has low sensitivity to C_d . Damping of mode λ_{3-4} has high sensitivity to C_d but has low sensitivity to L_s . Damping of both modes is highly sensitive to K_{pv} and K_{iv} .

For the mode frequencies, when the number of vehicles, L_s , K_{pv} and K_{pi} change, the frequencies of these two modes change only slightly. When K_{iv} changes in the range of 62% to 115% of the original value, the absolute value of imaginary parts of modes λ_{1-2} and λ_{3-4} change in the ranges of 19.951 to 27.008 and 19.928 to 27.438, respectively. When C_d changes in the range of 67% to 117% of the original value, the absolute value of imaginary parts of modes λ_{1-2} and λ_{3-4} change in the ranges of 30.892 to 23.366 and 31.245 to 23.627, respectively. Moreover, it is also worth noting that the smaller the C_d , the greater the influence on the mode frequencies. From this point of view, compared to K_{iv} , C_d has a greater impact on frequencies. Therefore, it can be concluded that, the frequencies of both modes are highly sensitive to C_d . K_{iv} of the voltage controller has a certain influence on the frequencies, which is a moderate influence.

Table VI shows the sensitivities of the two low-frequency modes to system parameters. If the system has been put into operation, from the perspective of minimum cost, the low-frequency oscillation can be suppressed by adjusting the controller parameters. For mode λ_{1-2} , during the design of traction power supply system, choosing the proper power supply line of the traction substation or the capacity of the

TABLE VI
SENSITIVITY COMPARISON OF THE LOW-FREQUENCY MODES

Parameter	Damping		Frequency	
	λ_{1-2}	λ_{3-4}	λ_{1-2}	λ_{3-4}
Number of vehicles	high	low	low	low
L_s	high	low	low	low
C_d	low	high	high	high
Voltage controller	K_{pv}	high	high	low
	K_{iv}	high	high	moderate
Current controller K_{pi}	low	low	low	low

traction transformer to reduce the equivalent impedance of the traction network, also can achieve the purpose of improving low-frequency stability.

V. NONLINEAR TIME-DOMAIN SIMULATION AND VERIFICATION

In order to verify the eigenvalue analysis results, nonlinear time-domain simulations are performed on the MATLAB/Simulink, and modal analysis based on the estimation of signal parameters via rotational invariance techniques (ESPRIT) is provided.

A. Waveforms As the Number of Vehicles Increases

The voltage waveforms, when the number of vehicles connected to the grid increases, are shown in Fig. 13. The system parameters are the same as those in Fig. 8. When 7 vehicles are connected to the grid, the amplitude fluctuations of the input voltage of on-board transformer (pantograph voltage) and the DC-link voltage of the units become stable around 1 s, as shown in Fig. 13(a).

When 8 vehicles are connected, the fluctuations of the pantograph voltage and the DC-link voltage also show a damped oscillation, and become stable around 3 s, as shown in Fig. 13(b). The oscillatory time is slightly longer than that when 7 vehicles are connected to the grid. However, when 9 vehicles are connected to the grid, the pantograph voltage and the DC-link voltage continuously oscillate. As shown in Fig. 13(c), the voltage amplitude exhibits a periodic oscillation.

In order to clearly compare with the eigenvalue analysis shown in Fig. 8 and Table III, this paper uses the ESPRIT method to perform modal identifications on the waveforms of the DC-link voltage. The ESPRIT method is based on the analysis of subspace signals and can effectively identify the parameters, such as the frequency and damping ratio, of each dominant mode of time-domain signal [33].

Table VII shows the modal analysis based on the ESPRIT method, using time-domain simulation waveforms of the DC-link voltage in Fig. 13. It can be concluded that, as the number of vehicles connected to the grid increases, the damping ratios and oscillatory frequencies decrease. Furthermore, when 9 vehicles are connected to the grid, the modal analysis shows that the oscillatory frequency is 3.87 Hz, and the damping ratio is -0.0009 , which are very close to the theoretical results given in Table III.

B. Voltage Waveforms As Control Parameters Change

Simulations with different control parameters when 9 vehicles are connected to the grid are conducted, and the results are shown in Fig. 14. Compared with the unstable results in Fig. 13(c), the system becomes stable when the voltage controller proportional gain K_{pv} is set to 0.12, as shown in Fig. 14(a), which is consistent to the eigenvalue analysis given in Fig. 9(a). Similarly, when the voltage controller integral gain K_{iv} is set to 12 or the current proportional gain K_{pi} is set to 0.06, the system also becomes stable, as shown in Fig. 14(b) and (c), which are consistent to the eigenvalue analysis respectively given in Fig. 9(b) and Fig. 10.

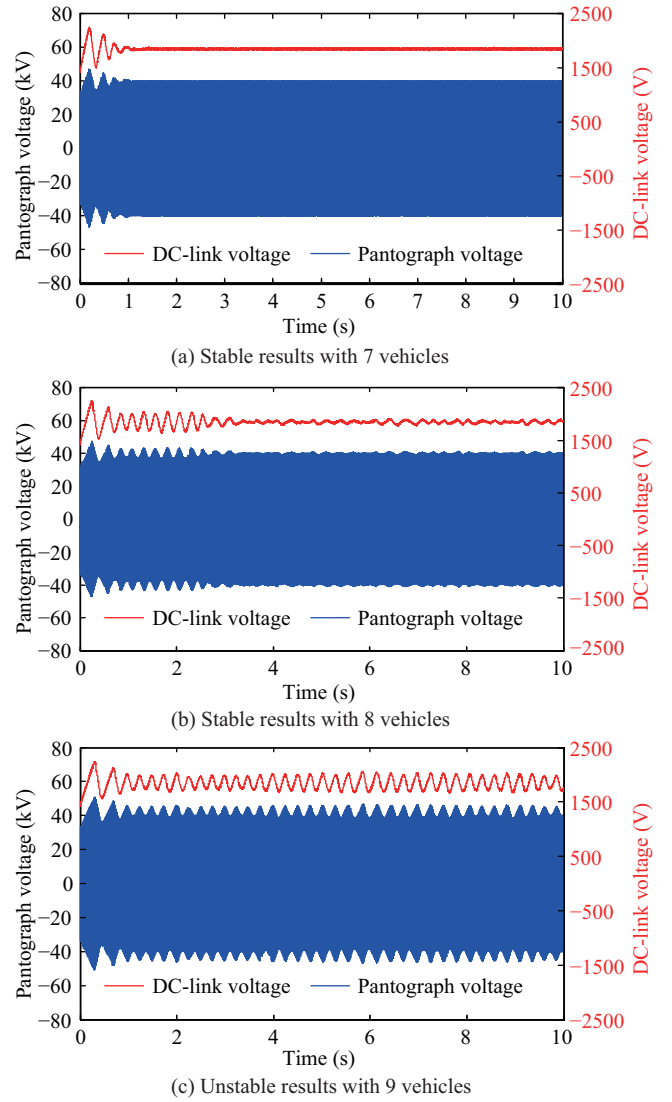


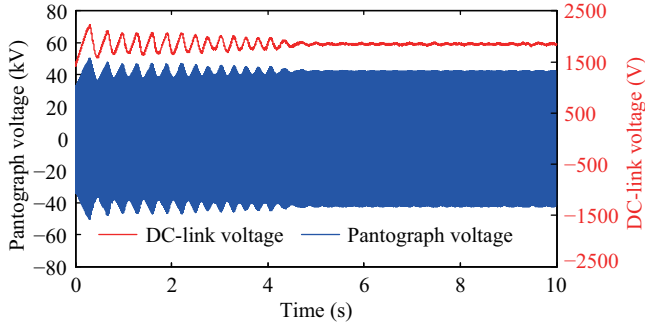
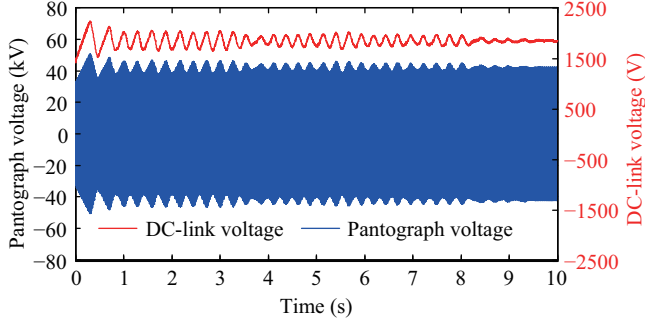
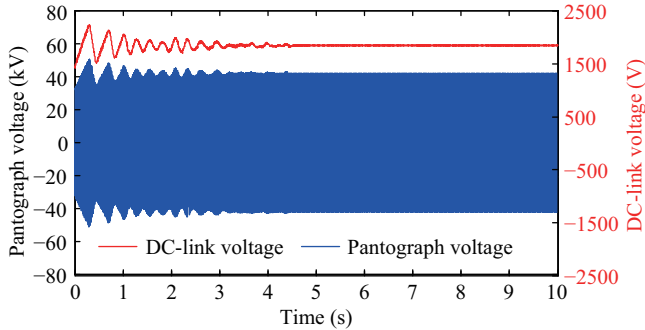
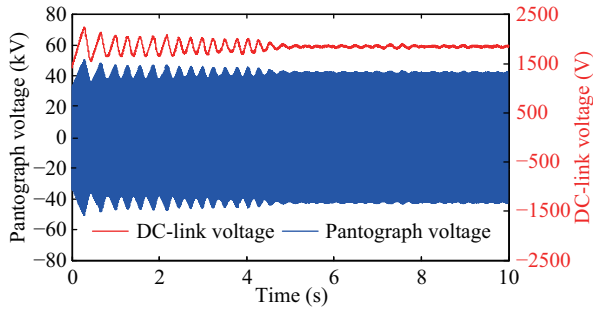
Fig. 13. The voltages of the pantograph and the DC-link when different numbers of vehicles are connected to the grid ($K_{pv} = 0.1$, $K_{iv} = 13$, $K_{pi} = 0.1$).

TABLE VII
MODAL ANALYSIS BASED ON ESPRIT METHOD

Case	Amplitude (V)	Frequency (Hz)	Damping ratio
7 vehicles	265	4.57	0.0806
8 vehicles	175	4.10	0.0037
9 vehicles	146	3.87	-0.0009

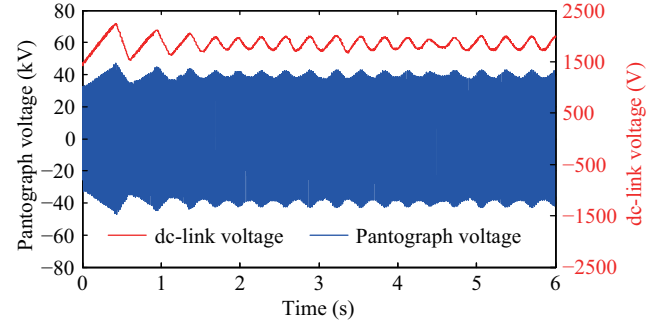
C. Waveforms As Grid Side Inductance and DC-link Capacitance Change

When the grid side inductance is reduced to 0.87 mH, the system is stable even if there are 9 vehicles connected to the grid, as shown in Fig. 15, and this is consistent with the eigenvalue analysis given in Fig. 11. When the DC-link capacitance increases to 16 mF, the system is still unstable, but the oscillatory frequency decreases to 3.25 Hz, as shown in Fig. 16. Therefore, increasing the DC-link capacitance will significantly reduce the oscillatory frequency, which is consistent with the eigenvalue analysis given in Fig. 12.

(a) Stable results with $K_{pv}=0.12$ ($K_{iv}=13$, $K_{pi}=0.1$)(b) Stable results with $K_{iv}=12$ ($K_{pv}=0.1$, $K_{pi}=0.1$)(c) Stable results with $K_{pi}=0.06$ ($K_{pv}=0.1$, $K_{iv}=13$)Fig. 14. Stable voltage waveforms when the controller parameters change ($P_{veh} = 9$).Fig. 15. Stable voltage waveforms with $L_s = 0.87$ mH ($K_{pv} = 0.1$, $K_{iv} = 13$, $K_{pi} = 0.1$) ($P_{veh} = 9$).

VI. CONCLUSION

This paper proposes the dynamic phasor model of the multiple vehicle-grid system in state-space form, then the eigenvalue-based method is introduced to analyze the LFO, and the conclusions are as follows.

Fig. 16. Unstable voltage waveforms with $C_d = 16$ mF ($K_{pv} = 0.1$, $K_{iv} = 13$, $K_{pi} = 0.1$) ($P_{veh} = 9$).

1) Two dominant low-frequency modes are identified by eigenvalues. The mode shape reveals that one of the modes involves the oscillation between the grid-connected converters and the traction network, while the other one involves the oscillation among these converters. The low-frequency oscillation caused by the increased number of the connected vehicles is equivalent to the system's unstable nature of a single converter connected to a weak grid.

2) For the low-frequency mode that is largely affected by the numbers of the vehicles, the state variables of the grid, the DC-link and the voltage controller have large participations. Taking into account all units connected to the grid, the sum of participation factors of the state variables of the DC-link or the voltage controller is hardly affected by the number of the vehicles. In addition, when the number of connected vehicles increases, the participation of the state variable of the grid increases.

3) The eigenvalue trajectories show that the two low-frequency modes have different sensitivities to control parameters, DC-link capacitance and grid side impedance. Moreover, the two low-frequency modes are both more sensitive to the voltage controller than to the current controller.

4) The nonlinear time-domain simulations have been performed, and the ESPRIT method has been applied to identify the oscillatory frequencies and damping ratios of the time-domain simulation signals. The simulations and ESPRIT analysis are both consistent with the theoretical results.

APPENDIX

Convert u_s , i_n , i_s , u_T and u_{abn} into xy rotating frame,

$$\begin{cases} u_{s_x} = 0 \\ u_{s_y} = -u'_s \end{cases} \quad (A1)$$

$$\begin{cases} i_{n_x} = -i'_n \sin \theta_n \\ i_{n_y} = -i'_n \cos \theta_n \end{cases} \quad (A2)$$

$$\begin{cases} i_{s_x} = -i'_s \sin \gamma \\ i_{s_y} = -i'_s \cos \gamma \end{cases} \quad (A3)$$

$$\begin{cases} u_{T_x} = -u'_T \sin \varphi_T \\ u_{T_y} = -u'_T \cos \varphi_T \end{cases} \quad (A4)$$

$$\begin{cases} u_{abn_x} = -m_n u_{dcn} \sin \delta_n \\ u_{abn_y} = -m_n u_{dcn} \cos \delta_n \end{cases} \quad (A5)$$

The derivatives of i_{n-x} , i_{n-y} , i_{s-x} and i_{s-y} with respect to time t are represented by:

$$\begin{cases} \frac{di_{n-x}}{dt} = -\sin \theta_n \frac{di'_n}{dt} - i'_n \cos \theta_n \frac{d\theta_n}{dt} \\ \frac{di_{n-y}}{dt} = -\cos \theta_n \frac{di'_n}{dt} + i'_n \sin \theta_n \frac{d\theta_n}{dt} \end{cases} \quad (A6)$$

$$\begin{cases} \frac{di_{s-x}}{dt} = -\sin \gamma \frac{di'_s}{dt} - i'_s \cos \gamma \frac{d\gamma}{dt} \\ \frac{di_{s-y}}{dt} = -\cos \gamma \frac{di'_s}{dt} + i'_s \sin \gamma \frac{d\gamma}{dt} \end{cases} \quad (A7)$$

Convert (12), (13) and (14) into xy rotating frame,

$$\begin{cases} L_T \frac{di_{n-x}}{dt} - \omega_1 L_T i_{n-y} + R_T i_{n-x} = u_{T-x} - u_{abn-x} \\ L_T \frac{di_{n-y}}{dt} + \omega_1 L_T i_{n-x} + R_T i_{n-y} = u_{T-y} - u_{abn-y} \end{cases} \quad (A8)$$

$$\begin{cases} L_s \frac{di_{s-x}}{dt} - \omega_1 L_s i_{s-y} + R_s i_{s-x} = u_{s-x} - u_{T-x} \\ L_s \frac{di_{s-y}}{dt} + \omega_1 L_s i_{s-x} + R_s i_{s-y} = u_{s-y} - u_{T-y} \end{cases} \quad (A9)$$

$$\begin{cases} u_{T-x} = R_g \left(i_{s-x} - 2 \sum_{n=1}^N i_{n-x} \right) \\ u_{T-y} = R_g \left(i_{s-y} - 2 \sum_{n=1}^N i_{n-y} \right) \end{cases} \quad (A10)$$

Substitute (A1)~(A7) into (A8)~(A10),

$$\begin{cases} L_T \left(\omega_0 i'_n \cos \theta_n - \sin \theta_n \frac{di'_n}{dt} - i'_n \cos \theta_n \frac{d\theta_n}{dt} \right) - R_T i'_n \sin \theta_n = -u'_T \sin \phi_T + m_n u_{dcn} \sin \delta_n \\ L_T \left(\omega_0 i'_n \sin \theta_n + \cos \theta_n \frac{di'_n}{dt} - i'_n \sin \theta_n \frac{d\theta_n}{dt} \right) + R_T i'_n \cos \theta_n = u'_T \cos \phi_T - m_n u_{dcn} \cos \delta_n \end{cases} \quad (A11)$$

$$\begin{cases} L_s \left(\sin \gamma \frac{di'_s}{dt} + i'_s \cos \gamma \frac{d\gamma}{dt} \right) = L_s \omega_1 i'_s \cos \lambda - u'_T \sin \phi_T - R_s i'_s \sin \gamma \\ L_s \left(i'_s \sin \gamma \frac{d\gamma}{dt} - \cos \gamma \frac{di'_s}{dt} \right) = -u'_s + L_s \omega_1 i'_s \sin \lambda + u'_T \cos \phi_T + R_s i'_s \cos \gamma \end{cases} \quad (A12)$$

$$\begin{cases} u'_T \cos \varphi_T = R_g \left(i'_s \cos \gamma - 2 \sum_{n=1}^N i'_n \cos \theta_n \right) \\ u'_T \sin \varphi_T = R_g \left(i'_s \sin \gamma - 2 \sum_{n=1}^N i'_n \sin \theta_n \right) \end{cases} \quad (A13)$$

Substitute the sine expressions of i_n into (15), without considering the second harmonic component, DC-link voltage is expressed as:

$$C_d \frac{du_{dcn}}{dt} = m_n i'_n \cos(\delta_n - \theta_n) - i_d \quad (A14)$$

Substitute (23) into (25), and convert the resulting equation into xy rotating frame. Use phasor amplitude and phase as variables, the modulation signal is expressed as follows.

$$\begin{cases} u_{dcn} m_n^* \cos \delta_n^* = u'_T \cos \phi_T - \omega_1 L_T I_n^* \sin \phi_T \\ -R_T I_n^* \cos \phi_T - K_{pi} I_n^* \cos \phi_T + K_{pi} i'_n \cos \theta_n \\ u_{dcn} m_n^* \sin \delta_n^* = u'_T \sin \phi_T + \omega_1 L_T I_n^* \cos \phi_T \\ -R_T I_n^* \sin \phi_T - K_{pi} I_n^* \sin \phi_T + K_{pi} i'_n \cos \theta_n \end{cases} \quad (A15)$$

Linearize (A11) around the equilibrium point,

$$\begin{cases} L_T \left(\cos \theta_{n0} \frac{d\Delta i'_n}{dt} - I_{n0} \sin \theta_{n0} \frac{d\Delta \theta_n}{dt} \right) = -U_{T0} \sin \phi_{T0} \Delta \phi_T + (-L_T \omega_1 \cos \theta_{n0} + R_T \sin \theta_{n0}) I_{n0} \Delta \theta_n - M_{n0} \cos \delta_{n0} \Delta u_{dcn} + M_{n0} U_{dc0} \sin \delta_{n0} \Delta \delta_n - U_{dc0} \cos \delta_{n0} \Delta m_n + \cos \phi_{T0} \Delta u'_T - (R_T \cos \theta_{n0} + L_T \omega_1 \sin \theta_{n0}) \Delta i_n \\ L_T \left(-\sin \theta_{n0} \frac{d\Delta i'_n}{dt} - I_{n0} \cos \theta_{n0} \frac{d\Delta \theta_n}{dt} \right) = -U_{T0} \cos \phi_{T0} \Delta \phi_T + (L_T \omega_1 \sin \theta_{n0} + R_T \cos \theta_{n0}) I_{n0} \Delta \theta_n + M_{n0} \sin \delta_{n0} \Delta u_{dcn} + M_{n0} U_{dc0} \cos \delta_{n0} \Delta \delta_n + U_{dc0} \sin \delta_{n0} \Delta m_n - \sin \phi_{T0} \Delta u'_T + (R_T \sin \theta_{n0} - L_T \omega_1 \cos \theta_{n0}) \Delta i_n \end{cases} \quad (A16)$$

Linearizing (A12) around the equilibrium point,

$$\begin{cases} L_s \left(\cos \gamma_0 \frac{d\Delta i'_s}{dt} - I_{s0} \sin \gamma_0 \frac{d\Delta \gamma}{dt} \right) = (-R_s \cos \gamma_0 - L_s \omega_1 \sin \gamma_0) \Delta i'_s + \Delta u'_s - \cos \varphi_{T0} \Delta u'_T + U_{T0} \sin \varphi_{T0} \Delta \varphi_T - (L_s \omega_1 \cos \gamma_0 - R_s \sin \gamma_0) I_{s0} \Delta \gamma \\ L_s \left(-\sin \gamma_0 \frac{d\Delta i'_s}{dt} - I_{s0} \cos \gamma_0 \frac{d\Delta \gamma}{dt} \right) = (R_s \sin \gamma_0 - L_s \omega_1 \cos \gamma_0) \Delta i'_s + \sin \varphi_{T0} \Delta u'_T + U_{T0} \cos \varphi_{T0} \Delta \varphi_T + (L_s \omega_1 \sin \gamma_0 + R_s \cos \gamma_0) I_{s0} \Delta \gamma \end{cases} \quad (A17)$$

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